Dynamic General Equilibrium Model

1. Benevolent Central Planner

- Suppose that, at each time t, a representative economic "agent" chooses how much to consume (c_t) and how much to save (k_t).
- The agent's utility function is given by

$$\sum_{t=0}^{\infty} \beta^t \ln c_t$$

where $\beta \in (0, 1)$ denotes a discount factor.

- Note that the agent does not derive utility from saving. Why, then, would he save? Because savings today are used to produce stuff tomorrow via the production function $f(k_{t-1}) = k_{t-1}^{\sigma}$. (Note: this production function is Cobb-Douglas, where the labor input is normalized to 1.)
- Therefore, the aggregate resource constraint at each time t is given by

$$c_t + k_t = k_{t-1}^{\sigma}$$

(Note: this is simply $C + I + G = Y$, without the G.)

- Now consider a benevolent central planner, whose task is to maximize $\sum_{t=0}^{\infty} \beta^t \ln c_t$ subject to $c_t + k_t = k_{t-1}^{\sigma}$.
- We wish to find c_t and k_t that maximize the agent's utility.

- *(Here's how to solve. Ignore if you wish.)*
- $\mathcal{L} = \sum \beta^t \left\{ \ln c_t + \lambda_t \left(k_{t-1}^{\sigma} - c_t - k_t \right) \right\}$

- $\mathcal{L}_c = \beta^t \frac{1}{c_t} - \beta^t \lambda_t = 0$
- $\mathcal{L}_k = \beta^{t+1} \lambda_{t+1} \sigma k_t^{\sigma-1} - \beta^t \lambda_t = 0$
- Combining these two, $\frac{1}{c_t} = \sigma\beta \frac{1}{c_{t+1}} k_t^{\sigma-1}$, which just says that the agent equates the marginal utility of consumption across all time periods
- Rewrite this as $\frac{1}{c_t} = \sigma\beta \frac{1}{c_{t+1}} \frac{k_t^\sigma}{k_t}$
- Rewrite this as $\frac{k_t}{c_t} = \sigma\beta \frac{1}{c_{t+1}} k_t^\sigma$
- Noting that, according to the resource constraint, $k_t^\sigma = c_{t+1} + k_{t+1}$, rewrite this as $\frac{k_t}{c_t} = \sigma\beta \frac{c_{t+1}+k_{t+1}}{c_{t+1}}$
- Now add $\frac{c_t}{c_t}$ to both sides and get $\frac{c_t+k_t}{c_t} = 1 + \sigma\beta \frac{c_{t+1}+k_{t+1}}{c_{t+1}}$
- Let $z_t = \frac{c_t+k_t}{c_t}$, rewrite as $z_t = 1 + \sigma\beta z_{t+1}$
- Solve forward: $z_t = 1 + \sigma\beta\left(1 + \sigma\beta\left(1 + \sigma\beta\left(\cdots\right)\right)\right) = 1 + \sigma\beta + (\sigma\beta)^2 + (\sigma\beta)^3 + \ldots$
- And because $\sigma\beta < 1$, this simplies to $z_t = \frac{1}{1-\sigma\beta}$
- Substituting for z_t, $\frac{c_t+k_t}{c_t} = \frac{1}{1-\sigma\beta}$, which means that $c_t^* = (1-\sigma\beta)(c_t + k_t) = (1-\sigma\beta) k_{t-1}^\sigma$
- Plug this back into the aggregate resources constraint and get $k_t^* = \sigma\beta k_{t-1}^\sigma$

- The solution is

$$c_t^* = (1-\sigma\beta) k_{t-1}^\sigma$$
$$k_t^* = \sigma\beta k_{t-1}^\sigma$$

- This means that, at each time t, the central planner should give $(1-\sigma\beta)$ of the pie to the agent to consumer, and the planner should save $\sigma\beta$.

2. Circular Flow Problem

- But what if there is no central planner to divide the pie?

2.1. The Firm

- The firm maximizes profit, $\pi = k_{t-1}^{\sigma} - r_t k_{t-1}$
- Profits are maximized where

$$\sigma k_{t-1}^{\sigma-1} = r_t$$

2.2. The Consumer

- The consumer maximizes $\sum_{t=0}^{\infty} \beta^t \ln c_t$ subject to the budget constraint $c_t + k_t = r_t k_{t-1}$
- $\mathcal{L} = \sum \beta^t \left\{ \ln c_t + \lambda_t \left(k_{t-1}^{\sigma} - c_t - k_t \right) \right\}$
- $\mathcal{L}_c = \beta^t \frac{1}{c_t} - \beta^t \lambda_t = 0$
- $\mathcal{L}_k = \beta^{t+1} \lambda_{t+1} r_{t+1} - \beta^t \lambda_t = 0$
- Combine these two to get $\frac{1}{c_t} = \beta \frac{1}{c_{t+1}} r_{t+1}$

2.3. Combine the Firm's and Consumer's solutions

- $\frac{1}{c_t} = \sigma \beta \frac{1}{c_{t+1}} k_t^{\sigma-1}$
- Same as the central planner's problem!